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DYNAMICS OF TROJAN POINT SATELLITES

EDWARD RAYMOND BRISTOL, JR.

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DYNAMICS OF TROJAN POINT SATELLITES

by

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B. Eng., Yale University, 1950

B.S., U.S. Naval Postgraduate School, 1959

Submitted in Partial Fulfillment
of the Requirements for the
Degree of Master of Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
June, 1960

DYNAMICS OF TROJAN POINT SATELLITES

by

Edward R. Bristol, Jr.

Submitted to the Department of Aeronautics and Astronautics on May 21, 1960 in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The problem of predicting the motion of a space vehicle in the vicinity of a Trojan point in cislunar space is considered. Specifically, it is desired to know if the vehicle will remain in this vicinity, or diverge from the Trojan point. If the equations of motion of the vehicle are linearized at the Trojan point, the solution of these equations indicates that the vehicle will assume a periodic motion about the Trojan point. Fundamentally, the problem reduces to determining the validity, or range of validity of these solutions.

Various methods are discussed along with their associated problems. Further investigation is proposed along the lines of identifying and reducing computational errors. Also it is proposed to investigate the significance of linearizing the equations of motion at points displaced from the Trojan point.

Thesis Supervisor: Paul E. Sandorff

Title: Associate Professor of
Aeronautics and Astronautics

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The graduate work for which this thesis is a partial requirement was performed while the author was assigned from the U.S. Naval Postgraduate School for graduate training at the Massachusetts Institute of Technology.

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LIST OF SYMBOLS

$A_1, B_1, C_1, A_2, B_2, C_2, D_2$	Constants of integration of the linearized equations of motion, equations 5.1 and 5.2
a,b	Coordinates of a point in the vicinity of a Trojan point, Section 8.1
B	Barycenter of earth-moon system, Fig. 3.1
C	The Jacobian constant, Eq.4.12
G	Gravitational Constant
$\bar{f}$	Resultant specific force on a vehicle, Fig. 4.1
$\bar{f}_1, \bar{f}_2$	Gravitational specific forces on a vehicle, Fig. 4.1
$\bar{f}_c$	Centrifugal specific force on a vehicle, Fig. 4.1
k	Scale factor, equations 5.15 through 5.22
k_1, k_2, k_3, k_4, k_5	Defined by equations 8.7 through 8.11
m_1	Mass of earth
m_2	Mass of moon
m_3	Mass of vehicle
p	Differential operator, $\frac{d}{dt}$
p_1, p_2	Angular frequencies of vibration, equations 5.1 and 5.2
r	Distance of vehicle from barycenter, Figure 3.1 or 4.1

r_1	Distance of vehicle from earth, Figure 3.1 or 4.1
r_2	Distance of vehicle from moon, Figure 3.1 or 4.1
$R_1 R_2$	Distances from the point (a,b) to the earth and moon respectively, equations 8.3 and 8.4
t	Dimensionless time, Section 4.1
u	Jacobian function, Eq. 4.5
v	Velocity of the vehicle in the x, y coordinatesystem: $v^2 = \dot{x}^2 + \dot{y}^2$
x, y	Coordinates in rotating earth-moon system, Fig. 4.1
z	Position of the vehicle in the complex plane, $z = x + iy$, Section 4.3
β	See Fig. 4.1
γ	See Fig. 3.1 or 4.1
ϵ	Distance of earth from barycenter, Section 3.1
θ	Polar angle in polar coordinates, Appendix B
λ	Distance of moon from barycenter, Section 3.1
ρ	Polar distance in polar coordinates, Appendix B
ϕ	See Figure 3.1 or 4.1
χ	See Figure 3.1
ψ	See Figure 3.1
Ω	Angular velocity of the earth-moon system with respect to inertial space

Superscripts

'	Quantities of the linearized model of the system
*	"Residual" quantities; the deviation of quantities of the actual system from those of the linearized model.

OBJECT

It is desired to study the orbital stability of a space vehicle at a Trojan point in cislunar space, i.e., where the earth, moon and vehicle form an equilateral triangle in the orbital plane of the earth and moon. If a vehicle originally placed at the Trojan point is indeed in a stable orbit relative to the Trojan point, it should then be determined what order of inaccuracies of injection can be tolerated to still establish a stable orbit.

This assumes that there is an operational requirement for a space vehicle of this sort which maintains its relative position with respect to the earth and moon. The requirement might be for a space navigation or communications satellite, or it might be for a replenishment station for a manned deep space probe. However, the specific advantages of a Trojan point orbit were not investigated further.

CHAPTER I

INTRODUCTION

One solution of the classical problem of three bodies is the equilateral triangle solution, discovered by Lagrange in 1772, where the three bodies are each equidistant from the other two and, in the absence of external perturbations, maintain this relative position. In the restricted three-body problem where the mass of the third body is small enough not to influence the motion of the other two bodies, the two equilateral triangle points in the plane of rotation of the two major bodies are called Trojan points. The Trojan asteroids, the first of which was discovered in 1906, have been found to remain in the vicinity of the equilateral triangle points with respect to Jupiter and the sun.

The term, orbital stability, is used in the sense that a body in a stable orbit in the vicinity of a Trojan point remains indefinitely in this vicinity, i.e., there is an upper bound to its distance from the Trojan point which does not increase with time.

If this definition is accepted literally, we must also limit the magnitude of the upper bound, since any

cislunar vehicle with less than escape velocity will remain in cislunar space, barring collisions with another body, and will have an upper bound to its displacement from any point in the earth-moon coordinate system. A convenient restriction on the rigorous definition of stability is not immediately apparent. However, the distinction is perhaps overly meticulous. We are concerned with a vehicle which remains in the vicinity of a Trojan point and, due to initial displacement from the Trojan point, initial non-zero relative velocity with respect to the Trojan point, or pertubations external to the system, appears to oscillate about the Trojan point. If it continues to do this, it is stable; if it diverges from the Trojan point, it is unstable.

Historically, the Trojan points have been considered stable for "small disturbances" based on a linearization of the equations of motion of the third body at the Trojan point. However, the stability of a body in the actual non-linear system has not been conclusively determined. In fact, there are arguments indicating that, even granted exact knowledge of the physical constants and unlimited precision and speed of digital computation, the stability can still not be established with mathematical rigor.

The existence of the Trojan asteroids is not as good an argument for stability as it seems at first thought. According to the solution of their linearized equations of motion, the periods of their two modes of vibration are 11.86 years and

467 years. The corresponding periods in cislunar space are 28.6 days and 91.7 days. Thus the 54 years available for observation of the Trojan asteroids corresponds, in one sense, to only about 11 days observation of a cislunar Trojan point satellite.

We would like to know specifically the ranges of initial conditions and magnitudes of perturbations for which a Trojan point vehicle would be stable. However, the study described here does not reveal such a criterion. Rather, it seems we must be content with specifying a time limit during which the vehicle will not exceed a specified displacement from the Trojan point. The intent of this investigation was to provide answers to questions of the sort, "What are the guidance accuracy requirements for establishing a cislunar Trojan point space station to remain within 20,000 miles of the Trojan point for at least 10 years?" Again, it must be admitted that the answer is not obtained in the present investigation. However, the directions toward which such answers lie may be indicated.

CHAPTER 2

RESTRICTIONS ON THE PROBLEM

The problem is considered to be that of the restricted three-body problem, where the mass of the third body, the vehicle, is negligible compared with the other two bodies. The vehicle will then have no effect on the motion of the other two bodies. This is obviously valid for the problem of earth, moon and a vehicle launched from the earth.

Another restriction is the elimination of all perturbative forces external to the earth-moon system. This is equivalent to considering the barycenter of the earth-moon system as an inertial point, and neglecting the curvature and velocity variations of its orbital motion. The perturbative effect of the sun is perhaps essential in the solution of the physical problem. However, it can later be added to the differential equations as a forcing function when the simpler system is more fully understood. Other perturbative forces such as that due to Jupiter, tidal forces, earth oblateness and radiation pressure are probably indeed negligible.

Also the orbit of the moon about the earth, or, more

precisely, about the barycenter of the earth-moon system, is considered to be circular. This effect can also be added later if desired.

Errors due to these simplifying assumptions will be discussed further in the text. Their effect is also considered by Bucheim (1).

CHAPTER 3

THE LIBRATION POINTS

3.1 The Geometry of a Space Vehicle Attracted by Two Bodies

In Fig. 3.1, the vehicle, m_3, is attracted by the two celestial bodies, m_1 and m_2, which revolve about their inertial barycenter, B. In cislunar space, m_1 and m_2 are the earth and moon, and, for convenience, they will be so named. The mass of the vehicle is considered small enough to have negligible effect on the motion of the earth and moon.

To simplify the equations, all distances are normalized with respect to the earth-moon distance. Distances from earth and moon to their barycenter are then ϵ and λ respectively where

$$\frac{\epsilon}{\lambda} = \frac{m_2}{m_1}$$

and

$$\epsilon + \lambda = 1$$

The gravitational forces on the vehicle are represented by $\bar{F}_1$ and $\bar{F}_2$ which are vector specific forces, or force per unit mass of the vehicle.

$$f_1 = \frac{Gm_1}{r_1^2} = \frac{G(m_1+m_2)}{r_1^2}\left(\frac{m_1}{m_1+m_2}\right) = \lambda\frac{G\,(m_1+m_2)}{r_1^2} \qquad (3.1)$$

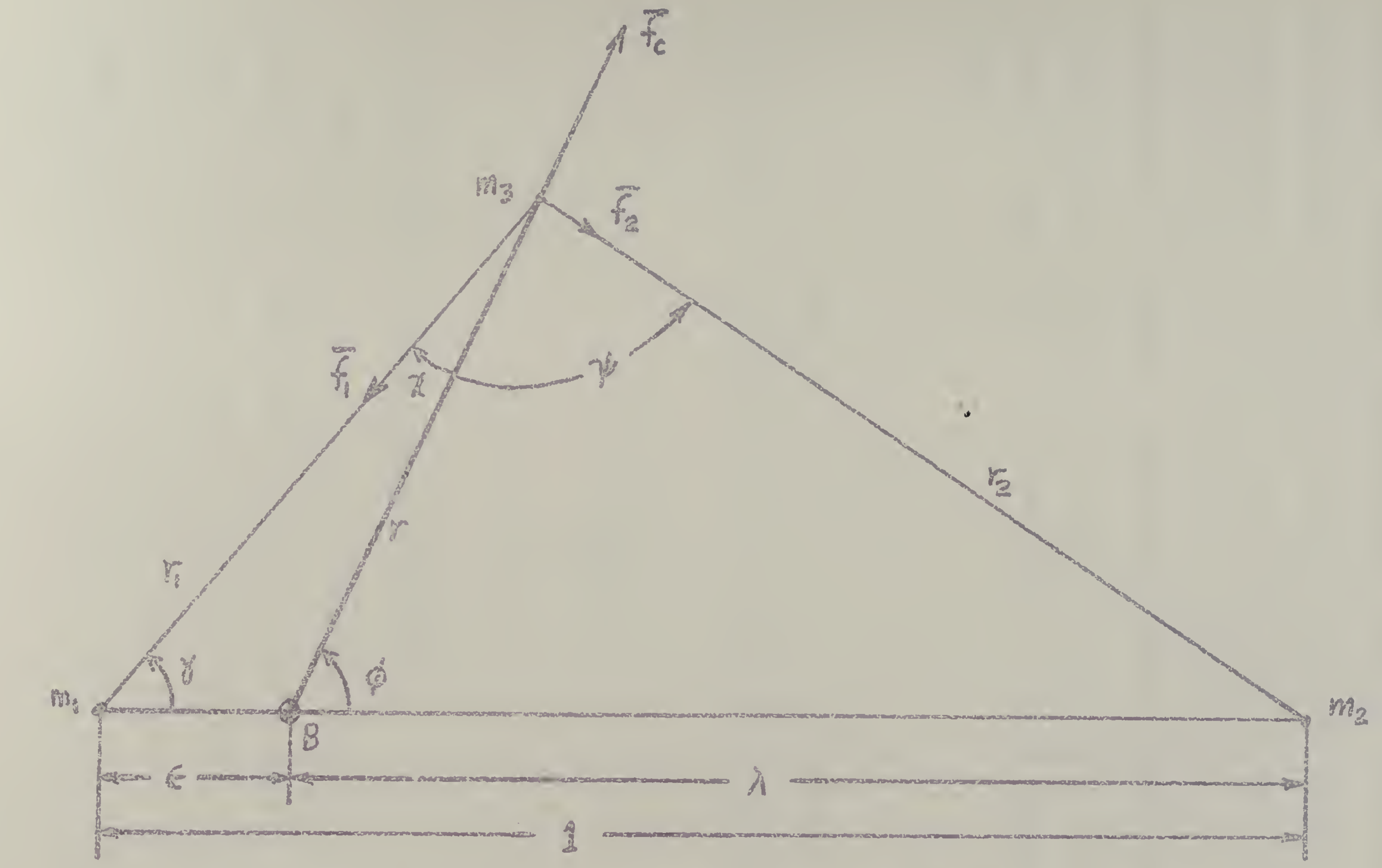


Fig. 3.1. Geometry of a Space Vehicle Attacted by Two Bodies

$$f_2 = \frac{Gm_2}{r_2^2} = \frac{G\ (m_1+m_2)}{r_2^2}\left(\frac{m_2}{m_1+m_2}\right) = \frac{\mu G(m_1+m_2)}{r_2^2} \qquad (3.2)$$

It is convenient to choose a rotating coordinate system in which the earth and moon appear stationary. The equations of motion in this coordinate system must then include centripetal and Coriolis accelerations resulting from the coordinate transformation. It is also convenient to consider the centripetal acceleration of the vehicle due to coordinate transformation as a centrifugal specific force, $\bar{f}_c$, along the radius vector from the barycenter.

$$f_c = r\Omega^2$$

where Ω is the angular velocity of the coordinate system with respect to inertial space. Then from Kepler's third law

$$\Omega^2 = \frac{G\ (m_1+m_2)}{(l)^3} \qquad (3.3)$$

$$f_c = G\ (m_1+m_2)\ r$$

3.2 The Existence of Libration Points

The following development, due to Klemperer and Benedikt, (2) shows the existence of five points in the plane of the moon's orbit at which a vehicle with zero relative velocity in the rotating coordinate system experiences zero resultant force. These points were

first discovered by Lagrange in 1772 as particular solutions of the three-body problem. Their existence could be proved with less effort by simple substitution of their coordinates into the equations of motion, but the following development also shows that these are the only five points at which this condition occurs.

Consider the vehicle to be at rest in the rotating system, With zero relative velocity, the Coriolis acceleration is zero and, if the vehicle is in equilibrium

$$\bar{F}_1 + \bar{F}_2 + \bar{F}_c = 0$$

Consider first the circumferential equilibrium of the vehicle, or the summation of force components perpendicular to the radius vector from the barycenter.

$$-f_1 \sin \chi + f_2 \sin \psi = 0$$

$$-\frac{\lambda G\ (m_1+m_2)}{r_1^2} \sin \chi + \frac{\epsilon G\ (m_1+m_2)}{r_2^2} \sin \psi = 0 \quad (3.4)$$

$$\frac{\lambda}{r_1^2} \sin \chi = \frac{\epsilon}{r_2^2} \sin \psi$$

One locus of circumferential equilibrium is the straight line through m_1 and m_2 where

$$\sin \chi = \sin \psi = 0$$

Also, by the law of sines

$$\frac{\sin \chi}{\epsilon} = \frac{\sin(\pi - \phi)}{r_1} = \frac{\sin \phi}{r_1}$$

$$\frac{\sin \psi}{\lambda} = \frac{\sin \phi}{r_2}$$

Substituting for sin χ and sin ψ in Eq. 3.4

$$\frac{\lambda(\epsilon \;\sin\phi)}{r_1^3} = \frac{\epsilon\;(\lambda\sin\phi)}{r_2^3}$$

And the other locus of circumferential equilibrium is the line

$$r_1 = r_2$$

Next consider the radial equilibrium of the vehicle, or the summation of force components parallel to the radius vector from the barycenter.

$$-f_1 \cos\chi - f_2 \cos\psi + f_c = 0$$

$$-\frac{\lambda G(m_1+m_2)}{r_1^2} \cos\chi - \frac{\epsilon G(m_1+m_2)}{r_2^2} \cos\psi + rG\,(m_1+m_2) = 0$$

$$-\frac{\lambda}{r_1^2} \cos\chi - \frac{\epsilon}{r_2^2} \cos\psi + r = 0 \qquad (3.5)$$

Finding cos χ and cos ψ by the law of cosines

$$\epsilon^2 = r^2 + r_1^2 - 2rr_1 \cos\chi$$

$$\cos\chi = \frac{r}{2r_1} + \frac{r_1}{2r} - \frac{\epsilon^2}{2rr_1}$$

$$\lambda^2 = r^2 + r_2^2 - 2rr_2 \cos\psi$$

$$\cos \psi = \frac{r}{2r_2} + \frac{r_2}{2r} - \frac{\lambda^2}{2rr_2}$$

Substituting for $\cos \chi$ and $\cos \psi$ in Eq. 3.5

$$-\frac{\lambda}{r_1^2}\left(\frac{r}{2r_1} + \frac{r_1}{2r} - \frac{\epsilon^2}{2rr_1}\right) - \frac{\epsilon}{r_2^2}\left(\frac{r}{2r_2} + \frac{r_2}{2r} - \frac{\lambda^2}{2rr_2}\right) + r = 0$$

Multiplying by $-2\epsilon r_1^3\, r_2^3$ to clear the fractions

$$\lambda r^2 r_2^3 + \lambda r_1^2 r_2^3 - \lambda \epsilon^2 r_2^3 + \epsilon r^2 r_1^3 + \epsilon r_1^3 r_2^2 - \epsilon \lambda^2 r_1^3 - 2r^2 r_1^3 r_2^3 = 0 \tag{3.6}$$

To eliminate r from the equation, r is found in terms of r_1 and r_2 by the law of cosines.

$$r^2 = \epsilon^2 + r_1^2 - 2\epsilon r_1 \cos \gamma \tag{3.7}$$

and also

$$r_2^2 = 1 + r_1^2 - 2r_1 \cos \gamma \tag{3.8}$$

Substituting for $r_1 \cos \gamma$ from Eq. 3.7 into Eq. 3.8

$$r^2 = \epsilon^2 + r_1^2 - \epsilon\ (1 + r_1^2 - r_2^2) = \epsilon^2 - \epsilon + r_1^2\ (1-\epsilon) + \epsilon r_2^2$$

$$r^2 = \epsilon(1-\epsilon) + r_1^2\ (1-\epsilon) + \epsilon r_2^2 = \epsilon\lambda + \lambda r_1^2 + \epsilon r_2^2$$

Substituting for r^2 in Eq. 3.6 and combining terms, the locus for radial equilibrium is obtained

$$\frac{2}{\epsilon} r_1^5 r_2^3 + \frac{2}{\lambda} r_1^3 r_2^5 - 2r_1^3 r_2^3 - r_1^5 - r_2^5 - \frac{1+\epsilon}{\lambda} r_1^3 r_2^2 - \frac{1+\epsilon}{\epsilon} r_1^2 r_2^3 +$$
$$+ r_1^3 + r_2^3 = 0$$

The equilibrium loci are shown in Fig. 3.2. The five

intersections of the two sets of curves are the only points at which a body would remain at equilibrium. These five points are statically stable in the rotating coordinate system, but the development yields no dynamic information. To specify the behavior of a vehicle in the vicinity of a libration point with deviations in position and non-zero relative velocity, the equations of motion of the vehicle must be investigated.

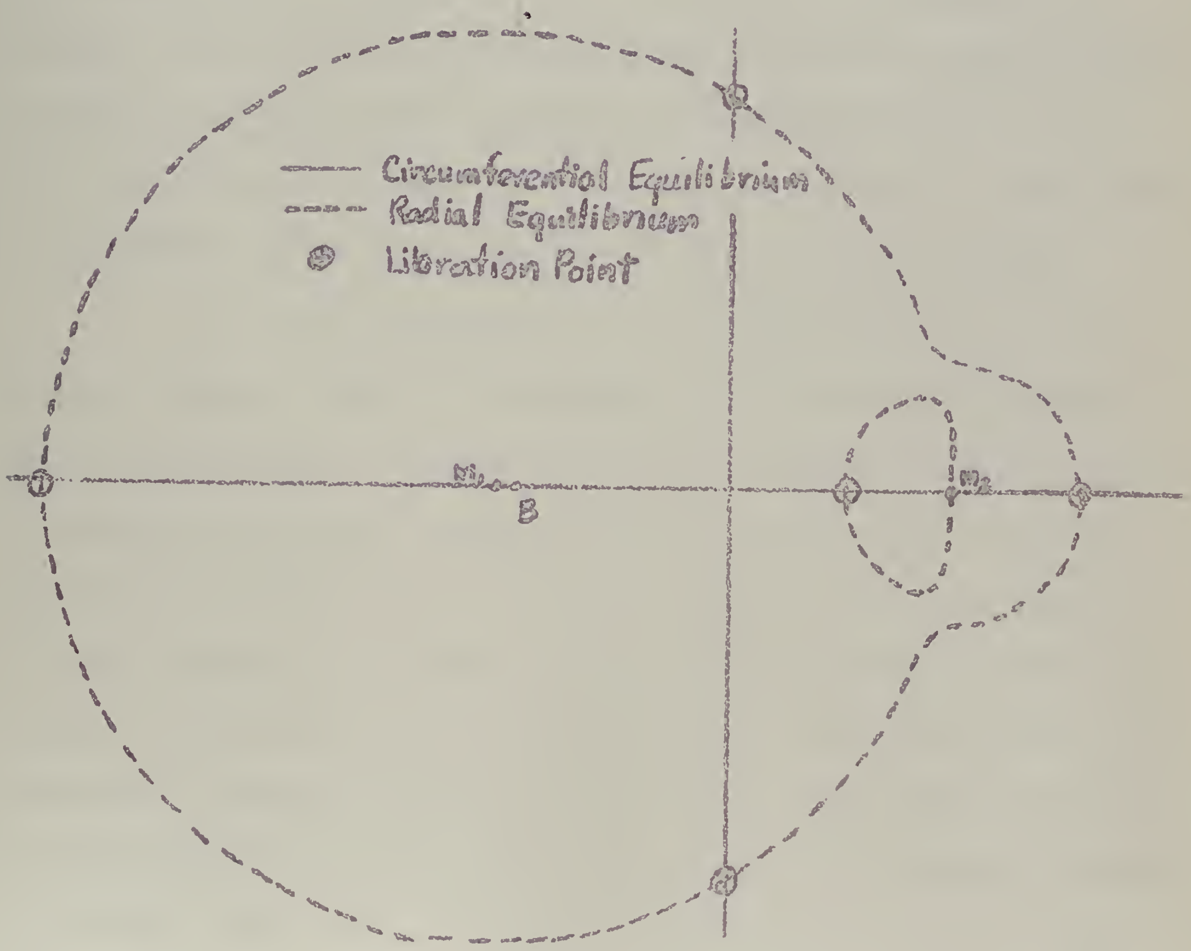


Fig. 3.2. Location of the Libration Points

CHAPTER 4

THE EQUATIONS OF MOTION

4.1 Formulation of the Equations of Motion

In an inertial coordinate system, the motion of the vehicle can be described by equating its acceleration to its gravitational specific force. Since it is desired to describe the motion in the coordinate set which rotates with the moon, the additional apparent accelerations which appear in the coordinate transformation must be included. The total acceleration consists of the sum of the following terms:

1. The acceleration of the coordinate system in inertial space. This is considered zero in the restricted three-body problem. In the physical problem in cislunar space, it corresponds to the acceleration of the earth-moon system in its orbit around the sun, including the perturbation effects of other planets. Its only significant component is the centripetal acceleration of the earth which seems to be the most appreciable effect neglected in confining the investigation to the restricted three-body problem. If later it appears desirable to consider this term, it can be included as a periodic forcing function in the equations of motion, representing the difference between the specific gravitational force of the sun on the

vehicle and that force on the barycenter of the earth-moon system.

2. The tangential acceleration. This is also considered to be zero since the moon is assumed to be in a uniform circular orbit about the earth. If the effect of the eccentricity of the moon's orbit were to be considered, this term would have to be included.

3. The centripetal acceleration. This is treated as a centrifugal specific force and grouped with the gravitational forces.

4. The relative acceleration. In the (x, y) rotating coordinate system, its components are $\ddot{x}$ and $\ddot{y}$ which are included in the equations of motion.

5. The Coriolis acceleration. Its magnitude is $2\Omega\sqrt{\dot{x}^2 + \dot{y}^2}$, and its direction is defined by the vector cross product of the system angular velocity and the relative velocity of the vehicle. Its x and y components are $-2\Omega\dot{y}$ and $+2\Omega\dot{x}$ respectively.

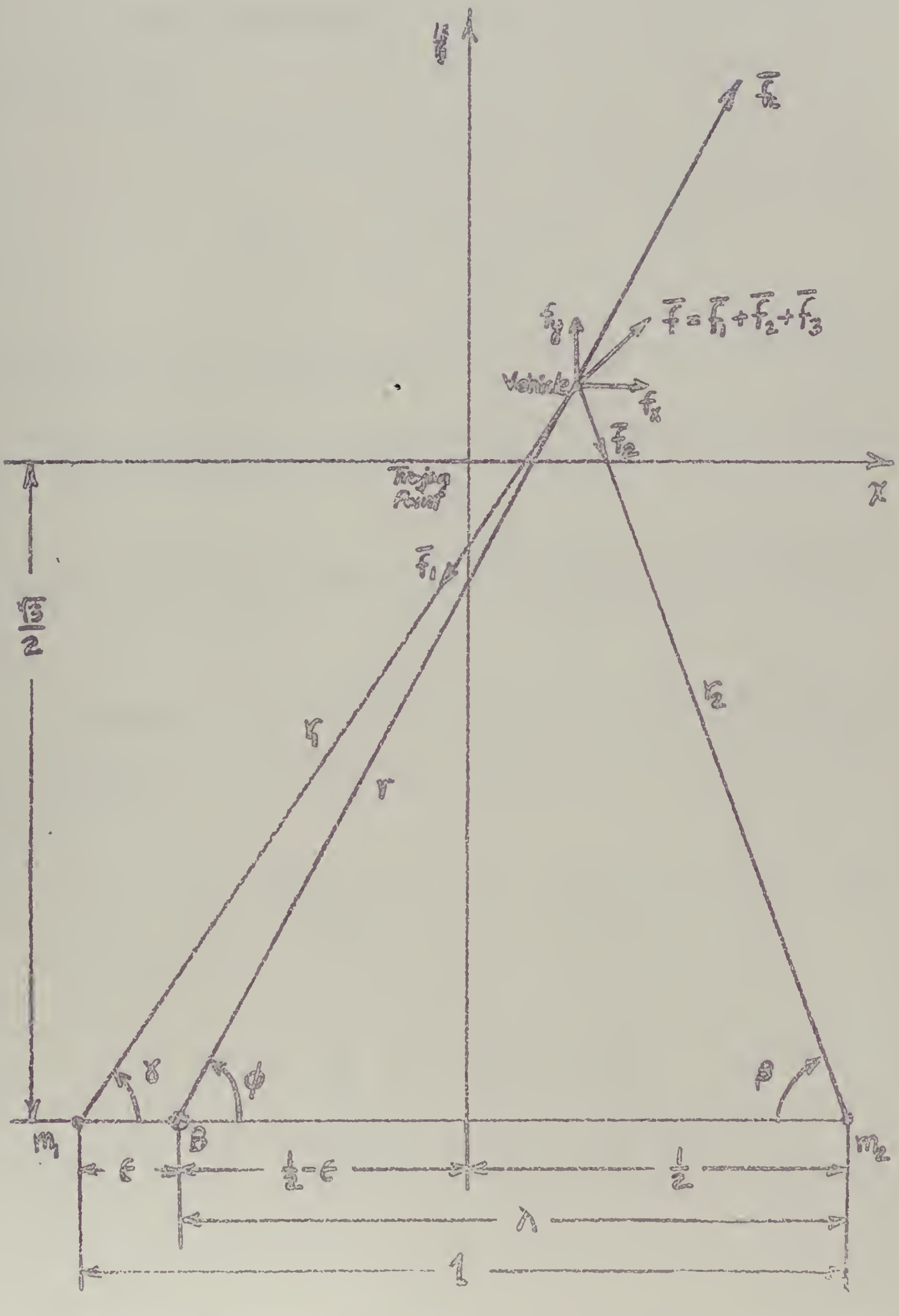


Fig. 4.1. Representation of the Rotating Coordinate Set.

As shown in Fig. 4.1, the origin of the coordinate system is chosen at an equilateral triangle point, or Trojan point, since we are primarily concerned with the dynamics of a vehicle in the vicinity of this point.

The x and y components of the vector equation of motion are then

$$\ddot{x} - 2\Omega\dot{y} = f_{1x} + f_{2x} + f_{cx}$$

$$\ddot{y} + 2\Omega\dot{x} = f_{1y} + f_{2y} + f_{cy}$$

A
Substituting from equations 3.1, 3.2, and 3.3 and resolving

$$\ddot{x} - 2\Omega\dot{y} = \frac{\lambda G(m_1+m_2)}{r_1^2}\cos\gamma + \frac{\epsilon G(m_1+m_2)}{r_2^2}\cos\beta + G(m_1+m_2)\, r\cos\phi$$

$$\ddot{y} + 2\Omega\dot{x} = \frac{\lambda G(m_1+m_2)}{r_1^2}\sin\gamma + \frac{\epsilon G(m_1+m_2)}{r_2^2}\sin\beta + G(m_1+m_2)\, r\sin\phi$$

To further simplify the equations, time will be normalized with respect to the period of rotation of the earth-moon system so that

$$t^2 = \frac{G(m_1+m_2)}{(1)^3}\, t'^2 \quad \text{where } t' \text{ is now real time}$$

and

$$\dot{x} = \Omega\,\frac{dx}{dt} = \sqrt{G(m_1+m_2)}\,\frac{dx}{dt}$$

$$\ddot{x} = \Omega^2\,\frac{d^2x}{dt'^2} = G(m_1+m_2)\,\frac{d^2x}{dt'^2}$$

with the dot denoting differentiation with respect to dimensionless time.

In dimensionless time, the equations of motion become

$$\dot{x} - 2\dot{y} = -\frac{\lambda}{r_1^2}\cos\gamma + \frac{\epsilon}{r_2^2}\cos\beta + r\cos\phi \quad (4.1)$$

$$\ddot{y} + 2\dot{x} = -\frac{\lambda}{r_1^2}\sin\gamma - \frac{\epsilon}{r_2^2}\sin\beta + r\sin\phi \quad (4.2)$$

Referring to Fig. 4.1, the quantities r_1, r_2, and the trigonometric functions of γ, β, and ϕ may be expressed in terms of x and y as follows:

$$r_1^2 = \left(y + \frac{\sqrt{3}}{2}\right)^2 + \left(x + \frac{1}{2}\right) = y^2 + \sqrt{3}y + x^2 + x + 1$$

$$r_2^2 = \left(y + \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2} - x\right)^2 = y^2 + \sqrt{3}y + x^2 - x + 1$$

$$r\cos\phi = x + \lambda - \frac{1}{2} = x + \frac{1}{2} - \epsilon$$

$$r\sin\phi = y + \frac{\sqrt{3}}{2}$$

$$\cos\gamma = \frac{x + \frac{1}{2}}{r_1} = \left(x + \tfrac{1}{2}\right)\left(y^2 + \sqrt{3}y + x^2 + x + 1\right)^{-\frac{1}{2}}$$

$$\cos\beta = \frac{\frac{1}{2} - x}{r_2} = -\left(x - \tfrac{1}{2}\right)\left(y^2 + \sqrt{3}y + x^2 - x + 1\right)^{-\frac{1}{2}}$$

$$\sin\gamma = \frac{y + \frac{\sqrt{3}}{2}}{r_1} = \left(y + \frac{\sqrt{3}}{2}\right)\left(y^2 + \sqrt{3}y + x^2 + x + 1\right)^{-\frac{1}{2}}$$

$$\sin \beta = \frac{y + \frac{\sqrt{3}}{2}}{r_2} = (y + \frac{\sqrt{3}}{2})\ (y^2 + \sqrt{3}y + x^2 - x + 1)^{-1/2}$$

Substituting in equations 4.1 and 4.2 the equations of motion become

$$\dot{x}-2\dot{y} = x+\tfrac{1}{2}-\epsilon- \quad (x+\tfrac{1}{2})(y^2+ 3y+x^2+x+1)^{-3/2} - \epsilon(x-\tfrac{1}{2})(y^2+\sqrt{3}y+x^2-x+1)^{-3/2} \tag{4.3}$$

$$\ddot{y} + 2\dot{x} = y+\frac{\sqrt{3}}{2} - \quad (y+\frac{\sqrt{3}}{2})(y^2+\sqrt{3}y+x^2+x+1)^{-3/2} - \epsilon(y+\frac{\sqrt{3}}{2})(y^2+\sqrt{3}y+x^2-x+1)^{-3/2} \tag{4.4}$$

4.2 Integration of the Equations of Motion

The only known integral of these equations was discovered by Jacobi. It has since been proved that, in this coordinate system, no other algebraic integrals exist.

$$\text{Let } u = -\epsilon x+\tfrac{1}{2}(y^2+\sqrt{3}y+x^2 x+1)+ \lambda (y^2+\sqrt{3}y+x^2+x+1)^{-1/2} + \epsilon (y^2+\sqrt{3}y+x^2-x+1)^{-1/2} \tag{4.5}$$

$$\text{Then } \frac{\partial u}{\partial x} = -\epsilon+x+\tfrac{1}{2}-\lambda(\tfrac{1}{2}+x)(y^2+\sqrt{3}y+x^2+x+1)^{-3/2} - \epsilon(x-\tfrac{1}{2})(y^2+\sqrt{3}y+x^2-x+1)^{-3}\ 2 \tag{4.6}$$

$$\frac{\partial u}{\partial y} = y + \frac{\sqrt{3}}{2} - \lambda (y+\frac{\sqrt{3}}{2})\ (y^2+\sqrt{3}y+x^2+x+1)^{-3/2} - \epsilon(y+\frac{3}{2})(y^2+\sqrt{3}y+x^2-x+1)^{-3/2} \tag{4.7}$$

and the equations of motion are

$$\ddot{x} - 2\dot{y} = \frac{\partial u}{\partial x} \tag{4.8}$$

$$\ddot{y} + 2\,\dot{x} = \frac{\partial u}{\partial y} \qquad (4.9)$$

Multiplying equations 4.8 and 4.9 by $2\dot{x}$ and $2\dot{y}$ respectively

$$2\,\ddot{x}\,\dot{x} - 2\,\dot{y}\,\dot{x} = 2\dot{x}\,\frac{\partial u}{\partial x} \qquad (4.10)$$

$$2\,\ddot{y}\dot{y} + 2\dot{y}\dot{x} = 2\dot{y}\,\frac{\partial u}{\partial y} \qquad (4.11)$$

Adding equations 4.10 and 4.11

$$2\ddot{x}\dot{x} + 2\,\ddot{y}\,\dot{y} = 2\dot{x}\,\frac{\partial u}{\partial x} + 2\dot{y}\,\frac{\partial u}{\partial y}$$

$$2\dot{x}\,\frac{d\dot{x}}{dt} + 2\dot{y}\,\frac{d\dot{y}}{dt} = 2\,\frac{\partial u}{\partial x}\,\frac{dx}{dt} + 2\,\frac{\partial u}{\partial y}\,\frac{dy}{dt}$$

Integrating

$$\dot{x}^2 + \dot{y}^2 = 2\int\left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y})dy\right) = 2u + C$$

$$C = -2u + \dot{x}^2 + \dot{y}^2$$

The Jacobi integral is

$$C = 2\epsilon x - (y^2+\sqrt{3}y+x^2-x+1) - 2\lambda(y^2+\sqrt{3}y+x^2-x+1)^{-\frac{1}{2}}$$
$$-2\epsilon(y^2+\sqrt{3}y+x^2-x+1)^{\frac{1}{2}} + \dot{x}^2 + \dot{y}^2 = \text{constant} \qquad (4.12)$$

4.3 Vectorial Representation of the Equations of Motion

It should be noted here that the subsequent equations apply to only one of the two Trojan points. For the other Trojan point, a symmetry about the x-axis applies, but the equations and directions may not be applied literally without consideration of this symmetry

If the motion of the vehicle is represented in the complex plane where

$$z = x + iy$$

equations 4.3 and 4.4 can be written as

$$\ddot{x} - 2\dot{y} = f_x \tag{4.13}$$

$$i\ddot{y} + 2i\dot{x} = if_y \tag{4.14}$$

Adding equations 4.13 and 4.14

$$\ddot{z} + 2i\dot{z} = f_x + if_y \tag{4.15}$$

At first thought, it would seem that, for a stable Trojan point orbit, the resultant of f_x and f_y should be directed toward the origin. This is not the case; it will be seen later that, in the vicinity of the Trojan point, f_x and f_y have the same sign as x and y respectively. Rather, it is the Coriolis acceleration which provides an average relative acceleration, $\ddot{z}$, toward the origin if the orbit is to be stable. The vectorial representation of Eq. 4.15 is shown in Fig. 4.2. This indicates that the orbit must have a gross motion clockwise about the origin for stability.

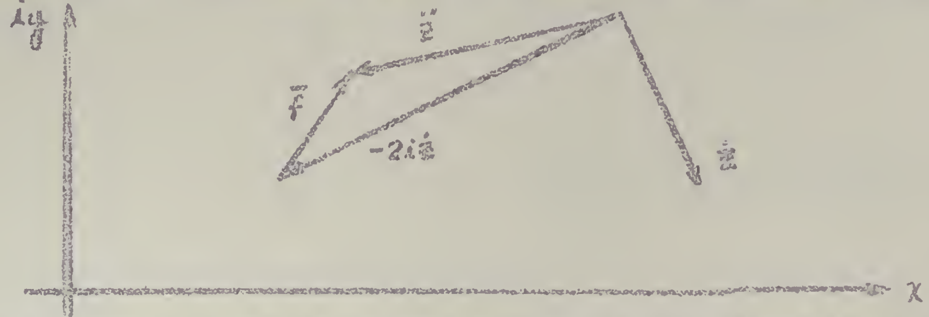


Fig. 4.2. Vectorial Representation of the Equations of Motion.

4.4 Linearization of the Equations of Motion

As a first order approximation of the behavior of a vehicle in the vicinity of a libration point, the equations of motion are linearized and the solution of the linearized equations is examined. The procedure used by Moulton (3) is essentially to linearize the right hand sides of equations 4.1 and 4.2 by Taylor series and solve the resulting linearized equations. Whittaker (4) uses a different approach, working with the equations of motion in the form of four first order differential equations, with the same results.

The first approach will be extended here to obtain a more complete linear model of the system, including an integral of the equations equivalent to the Jacobian. This will later be used for a more complete comparison of the actual and linearized systems.

Linearizing the differential equations is equivalent to retaining second order terms in the Taylor series representation of the Jacobian function, Eq. 4.5. Letting primed symbols represent the quantities of the linearized system Eq. 4.5 can be written as

$$u'(x,y) \; 2 \; u(0,0) + \frac{\partial u}{\partial x}(0,0)x + \frac{\partial u}{\partial y}(0,0)\,y + \tfrac{1}{2}\frac{\partial^2 u}{\partial x^2}(0,0)x^{\partial} + \qquad (4.16)$$

$$\frac{\partial^2 u}{\partial x \partial y}(0,0)xy + \tfrac{1}{2}\frac{\partial^2 u}{\partial y^2}(0,0)y^2$$

Substituting x = y = 0 in equations 4.5, 4.6 and 4.7

$$u\,(0,0) = \tfrac{1}{2} + \lambda + \epsilon = \frac{3}{2}$$

$$\frac{\partial u}{\partial x}\,(0,0) = -\epsilon + \tfrac{1}{2} - \tfrac{1}{2}\lambda + \tfrac{1}{2}\epsilon = 0$$

$$\frac{\partial u}{\partial y}\,(0,0) = \frac{\sqrt{3}}{2} - \lambda\frac{\sqrt{3}}{2} - \epsilon\,\frac{\sqrt{3}}{2} = 0$$

Differentiating equations 4.6 and 4.7

$$\frac{\partial^2 u}{\partial x^2} = 1 + 3\lambda(x+\tfrac{1}{2})^2(y^2+\sqrt{3}y+x^2+x+1)^{-5/2} - \lambda(y^2+\sqrt{3}y+x^2+x+1)^{-3/2}$$
$$+3\epsilon(x-\tfrac{1}{2})^2(y^2+\sqrt{3}y+x^2-x+1)^{-5/2} - \epsilon\,(y^2+\sqrt{3}y+x^2-x+1)^{-3/2} \tag{4.17}$$

$$\frac{\partial^2 u}{\partial y^2} = 1 + 3\lambda(y+\frac{\sqrt{3}}{2})^2(y^2+\sqrt{3}y+x^2+x+1)^{-5/2} - \lambda(y^2+\sqrt{3}y+x^2+x+1)^{-3/2}$$
$$+3\epsilon(y+\frac{\sqrt{3}}{2})^2(y^2+\sqrt{3}y+x^2-x+1)^{-5/2} - \epsilon\,(y^2+\sqrt{3}y+x^2-x+1)^{-3/2} \tag{4.18}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 3\lambda(x+\tfrac{1}{2})\,(y+\frac{\sqrt{3}}{2})\,(y^2+\sqrt{3}y+x^2+x+1)^{-5/2} + 3\epsilon(x-\tfrac{1}{2})(y+\frac{\sqrt{3}}{2})(y^2+\sqrt{3}y+x^2-x+1)^{-5/2} \tag{4.19}$$

Substituting x = y = 0 in equations 4.17, 4.18 and 4.19

$$\frac{\partial^2 u}{\partial x^2}\,(0,0) = 1 + \frac{3}{4}\lambda - \lambda + \frac{3}{4}\epsilon - \epsilon = \frac{3}{4}$$

$$\frac{\partial^2 u}{\partial y^2}\,(0,0) = 1 + \frac{9}{4}\lambda - \lambda + \frac{9}{4}\epsilon - \epsilon = \frac{9}{4}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{3\sqrt{3}}{4}\lambda - \frac{3\sqrt{3}}{4}\epsilon = \frac{3\sqrt{3}}{4}(\lambda - \epsilon) = \frac{3\sqrt{3}}{4}(1-2\epsilon)$$

Substituting these quantities in equation 4.16, the Jacobian function for the linearized model is

$$u' = \frac{3}{2} + \frac{3}{8} x^2 + \frac{3\sqrt{3}}{4} (1 - 2\epsilon) xy + \frac{9}{8} y^2 \qquad (4.20)$$

Differentiating, the linearized equations of motion are obtained

$$\ddot{x} - 2\dot{y} = \frac{\partial u'}{\partial x} = \frac{3}{4} x + \frac{3\sqrt{3}}{4} (1 - 2\epsilon) y \qquad (4.21)$$

$$\ddot{y} + 2\dot{x} = \frac{\partial u'}{\partial y} = \frac{3\sqrt{3}}{4} (1-2\epsilon) x + \frac{9}{4} y \qquad (4.22)$$

and the integral of the equations, equivalent to the Jacobian, is

$$C' = \dot{x}^2 + \dot{y}^2 - 3 - \frac{3}{4} x^2 - \frac{3\sqrt{3}}{2} (1-2\epsilon) xy - \frac{9}{4} y^2 \qquad (4.23)$$

CHAPTER 5

THE SOLUTION OF THE LINEARIZED EQUATIONS OF MOTION

5.1 The Stability Criterion

In differential operator form, equations 4.21 and 4.22 can be written as

$$\left(p^2 - \frac{3}{4}\right) x - \left[2p + \frac{3\sqrt{3}}{4}(1-2\epsilon)\right] y = 0$$

$$\left[2p - \frac{3\sqrt{3}}{4}(1-2\epsilon)\right] x + \left(p^2 - \frac{9}{4}\right) y = 0$$

The characteristic equation of this system is

$$\left(p^2 - \frac{3}{4}\right)\left(p^2 - \frac{9}{4}\right) + \left[2p + \frac{3\sqrt{3}}{4}(1-2\epsilon)\right]\left[2p - \frac{3\sqrt{3}}{4}(1-2\epsilon)\right] = 0$$

or

$$p^4 + p^2 + \frac{27}{4}\epsilon(1-\epsilon) = 0$$

The roots of the characteristic equation are

$$p^2 = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \frac{27}{4}\epsilon(1-\epsilon)}$$

If the second term is real, the characteristic equation yields two pairs of pure imaginary roots, and the solution consists of pure sinusoids. However, if the second term is imaginary, the roots have positive real parts, and the amplitude of the solution increases exponentially with time.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

$$[illegible]$$

[illegible]

The stability criterion is then

$$\tfrac{1}{4} - \frac{27}{4}\,\epsilon\,(1-\epsilon) > 0$$

or

$$\epsilon^2_{max} - \epsilon_{max} + \frac{1}{27} = 0$$

$$\epsilon_{max} = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \frac{1}{27}} = 0.03852$$

$$\left(\frac{\epsilon}{\lambda}\right)_{max} = \left(\frac{m_2}{m_1}\right)_{max} = 0.04006 = \frac{1}{24.96}$$

In the earth-moon system, $\frac{m_2}{m_1}$ is approximately $\frac{1}{81.3}$, and the stability criterion is met.

5.2 The Straight-Line Libration Points

Moulton uses a similar development to show that the straight-line libration points are unstable. Another presentation of this approach is made by Buchheim (5).

If the equations of motion are linearized about a straight-line libration point, the characteristic equation of the resulting system yields two imaginary roots and two real roots, one of which is always positive. A part of the solution must then increase exponentially with time. The system is thus considered unstable; a small disturbance in displacement from the libration point will increase with time.

5.3 The Trojan Point Solution

If the mass ratio $\frac{m_2}{m_1}$ is small enough, as it is in the earth-moon system, the linearized equations of motion yield a solution of the form

$$x = A_1 \sin p_1 t + B_1 \cos p_1 t + C_1 \sin p_2 t + D_1 \cos p_2 t \tag{5.1}$$

$$y = A_2 \sin p_2 t + B_2 \cos p_2 t + C_2 \sin p_2 t + D_2 \cos p_2 t \tag{5.2}$$

where

$$p_1^2 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 - 27\epsilon\ (1-\epsilon)}$$

$$p_2^2 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 27\epsilon\ (1-\epsilon)}$$

and four of the eight constants of integration are independent. The relationships among the constants can be determined by substituting the solution back into the linear differential equations. Solving for the constants of Eq. 5.2 in terms of the constants of Eq. 5.1

$$A_2 = \frac{-\sqrt{3}\ (1-2\epsilon)\ A_1 - 2p_1 B_1}{p_1^2 + \frac{9}{4}} \tag{5.3}$$

$$B_2 = \frac{2p_1 A_1 - \frac{3}{4}\sqrt{3}\,(1-2\epsilon)\,B_1}{p_1^2 + \frac{9}{4}} \qquad (5.4)$$

$$C_2 = \frac{-\frac{3}{4}\sqrt{3}\,(1-2\epsilon)\,C_1 - 2p_2 D_1}{p_2^2 + \frac{9}{4}} \qquad (5.5)$$

$$D_2 = \frac{2p_2\, C_1 - \frac{3}{4}\sqrt{3}\,(1-2\epsilon)\,D_1}{p_2^2 + \frac{9}{4}} \qquad (5.6)$$

The constants A_1, B_1, C_1, D_1 are then determined by the initial conditions of displacement and velocity, $x\ (0)$, $y\ (0)$, $\dot{x}\ (0)$ and $\dot{y}\ (0)$.

5.4 Elimination of One Mode of Oscillation

To better understand the nature of the solution that has been obtained from the linearized equations, it would be desirable to determine how the solution is divided between the two modes of vibration as a function of initial conditions. The constants of equations 5.1 and 5.2 can be determined as a function of initial conditions, in terms of the natural frequencies p_1 and p_2. This is done in Appendix A using numerical constants for the earth-moon system, but the physical process by which the system is excited to different proportions of the two modes of vibration is not readily apparent.

To obtain further insight along this line, we determine what specification of initial conditions will excite only one mode of vibration, say the p_1 mode. For simplicity we arbitrarily specify the initial displacement as

$$x\ (0) = 0 \tag{5.7}$$

$$y\ (0) = 0 \tag{5.8}$$

We are, of course, concerned with much smaller displacements, but the results can be scaled down. If we specify that the initial velocity is chosen to eliminate the p_2 mode, then equations 5.1 and 5.2 become

$$x = A_1 \sin p_1 t + B_1 \cos p_1 t$$

$$y = A_2 \sin p_1 t + B_2 \cos p_1 t$$

Substituting initial conditions

$$X\ (0) = B_1 = 0$$

$$y\ (0) = B_2 = 1$$

$$\dot{x}\ (0) = p_1 A_1$$

$$\dot{y}\ (0) = p_1 A_2$$

Also from Eq. 5.3

$$A_2 = \frac{-\frac{3}{4}\sqrt{3}\ (1-2\epsilon)\ A_1}{p_1^2 + \frac{9}{4}}$$

$$B_2 = \frac{2p_1 A_1}{p_1^2 + \frac{9}{4}}$$

Since $B_2 = 1$ then $A_1 = \frac{1}{2p_1}\left(p_1^2 + \frac{9}{4}\right)$

$$\text{and } A_2 = -\frac{1}{2p_1}\ \frac{3}{4}\sqrt{3}\ (1-2\epsilon)$$

Thus the initial velocities must be specified as

$$\dot{x}\ (0) = \tfrac{1}{2}\left(p_1^2 + \frac{9}{4}\right) \tag{5.9}$$

$$\dot{y}\ (0) = -\frac{3}{8}\sqrt{3}\ (1-2\epsilon) \tag{5.10}$$

And the subsequent motion of the particle is specified as

$$x = \frac{p_1^2 + \frac{9}{4}}{2p_1}\ \sin p_1 t \tag{5.11}$$

$$\frac{y = \frac{3}{4}\sqrt{3}\ (1-2\epsilon)}{y = \quad 2p_1}\ \sin p_1 t + \cos p_1 t \tag{5.12}$$

Solving for $\sin p_1 t$ and $\cos p_1 t$ in terms of x and y, we can then eliminate time by the equation

$$\left[\sin^2 p_1 t\right](x,y) + \cos^2 p_1 t\ (x,y) = 1$$

After simplification, the equation of the "orbit" about

the Trojan point is found to be

$$y^2 + \frac{3\sqrt{3}\ (1-2\epsilon)}{2(p_1^2 + \frac{9}{4})}\ xy + \left[\frac{27\ (1-2\epsilon)^2 + 4p_1^2}{16(p_1^2 + \frac{9}{4})^2}\right] x^2 = 1 \tag{5.13}$$

This is the equation of an ellipse with its center at the origin. It should be noted that this is not a Keplerian elliptical orbit. Angular momentum is not conserved in this coordinate system, and there is no central force field analogous to that of a Keplerian orbit in inertial space.

Although the motion of the vehicle about the Trojan point described in the relative coordinate system may not be correctly termed an orbit, the term will be retained for convenience.

If we let v be the magnitude of the velocity of the vehicle in the relative coordinate system, i.e.

$$v^2 = \dot{x}^2 + \dot{y}^2$$

Then this velocity can be found at any point on the orbit from Eq. 4.19

$$v^2 = C' + 3 + \frac{3}{4}\ x^2 + \frac{3}{2}\sqrt{3}\ (1-2\epsilon)\ xy + \frac{9}{4}\ y^2$$

C' is a constant which can be calculated from initial conditions

$$C' = v^2\ (0) - 3 - \frac{9}{4}$$

From equations 5.9 and 5.10

$$v^2\ (0) = \frac{1}{4}\ (p_1^2 + \frac{9}{4})^2 + \frac{27}{64}\ (1-2\epsilon)^2$$

and at any point on the orbit

$$v^2 = \frac{1}{4}\ (p_1^2 + \frac{9}{4})^2 + \frac{27}{64}\ (1-2\epsilon) - \frac{9}{4} + \frac{3}{4}\ x^2 + \frac{3}{2}\sqrt{3}\ (1-2\epsilon)\ xy + \frac{9}{4}\ y^2 \qquad (5.14)$$

Considering an arbitrary point on the orbit to be the initial point, the initial velocity necessary to excite only the p_1 mode is determined. Its magnitude is obtained from Eq. 5.14, and its direction is specified by the tangent to the orbit computed from Eq. 5.13.

If we now divide all displacement and velocity variables in equations 5.7 through 5.14 by an arbitrary scaling factor, we obtain a new orbit with the same center, shape and orientation, which is also a solution of the linearized equatimns with initial conditions specified to eliminate the p_2 mode. For a continuous variation of scaling factor,

we have an infinite family of ellipses covering the x-y plane.

$$x(0) =)\ 0 \tag{5.15}$$

$$y(0) = k \tag{5.16}$$

$$\dot{x}(0) = \frac{k}{2}\left(p_1^2 + \frac{9}{4}\right) \tag{5.17}$$

$$\dot{y}(0) = \frac{3k}{8}\sqrt{3}\,(1-2\epsilon) \tag{5.18}$$

$$x = \frac{k}{2p_1}\left(p_1^2 + \frac{9}{4}\right) \sin p_1 t \tag{5.19}$$

$$y = \frac{3k}{8p_1}\sqrt{3}\,(1-2\epsilon) \sin p_1 + k \cos p_1 t \tag{5.20}$$

$$y^2 + \left[\frac{3\sqrt{3}\,(1-2\epsilon)}{2\left(p_1^2 + \frac{9}{4}\right)}\right] xy + \left[\frac{27(1-2\epsilon)^2 + 4p_1^2}{16\left(p_1^2 + \frac{9}{4}\right)^2}\right] x^2 = k^2 \tag{5.21}$$

$$v^2 = \frac{k^2}{4}\left(p_1^2 + \frac{9}{4}\right)^2 + \frac{27k^2}{64}(1-2\epsilon)^2 - \frac{9k^2}{4} + \frac{3}{4}x^2 + \frac{3}{2}\sqrt{3}(1-2\epsilon)xy + \frac{9}{4}y^2 \tag{5.22}$$

Given any point in the x, y plane as the initial point,

we can calculate k by substitution in Eq. 5.21. We can then determine the required initial velocity vector to excite the p_1 mode only from equations 5.17 and 5.18 and the resulting orbit is specified by equations 5.19 through 5.22.

If we wish to eliminate the p_1 mode instead of the p_2 mode, the procedure is the same, and the resulting equations are identical with equations 5.7 through 5.23 with the substitution of p_2 for p_1.

5.5 Determination of the Orbit from Specified Initial Conditions

We now have the mathematical equipment for answering the question, "Given an arbitrary initial position, what vector initial velocity is required to excite only mode p_1 or p_2?" We do not have as convenient a physical representation for answering the question "Given an arbitrary initial position and an arbitrary initial velocity, what does the orbit look like?" The temptation is to substitute numbers into the equations of Appendix A. However, we can do it semi-graphically by a two stage trial and error method, and perhaps gain a better physical insight of the problem.

Let the equations of the orbit be written as

$$x = x_1 + x_2 \tag{5.23}$$

$$y = y_1 + y_2 \tag{5.24}$$

where $x_1 = A_1 \sin p_1 t + B_1 \cos p_1 t$

$y_1 = A_2 \sin p_1 t + B_2 \cos p_2 t$

$x_2 = C_1 \sin p_2 t + D_1 \cos p_2 t$

$y_2 = C_2 \sin p_2 t + D_2 \cos p_2 t$

To determine the orbit resulting from specified initial conditions, pick a point $\left[x_1(0),\ y_1(0)\right]$ as a trial. This determines an ellipse of the family of Eq. 5.21 and specifies the required $\dot{x}_1(0)$ and $\dot{y}_1(0)$. It also specifies the point $\left[x_2(0),\ y_2(0)\right]$ by equations 5.23 and 5.24. In turn, this point also specifies an ellipse of the family of Eq. 5.21 with p_2 substituted for p_1, and gives the required $\dot{x}_2(0)$ and $\dot{y}_2(0)$. If the trial is correct, the vector addition of $v_1(0)$ and $v_2(0)$ gives the specified initial condition $v(0)$, and the orbit can be constructed by the addition of a vector generating the p_2 ellipse at a relative angular velocity of $\frac{p_2}{p_1}$.

This method is illustrated in Fig. 5-1 which is not drawn to scale. It is included only for possible clarification of the above explanation.

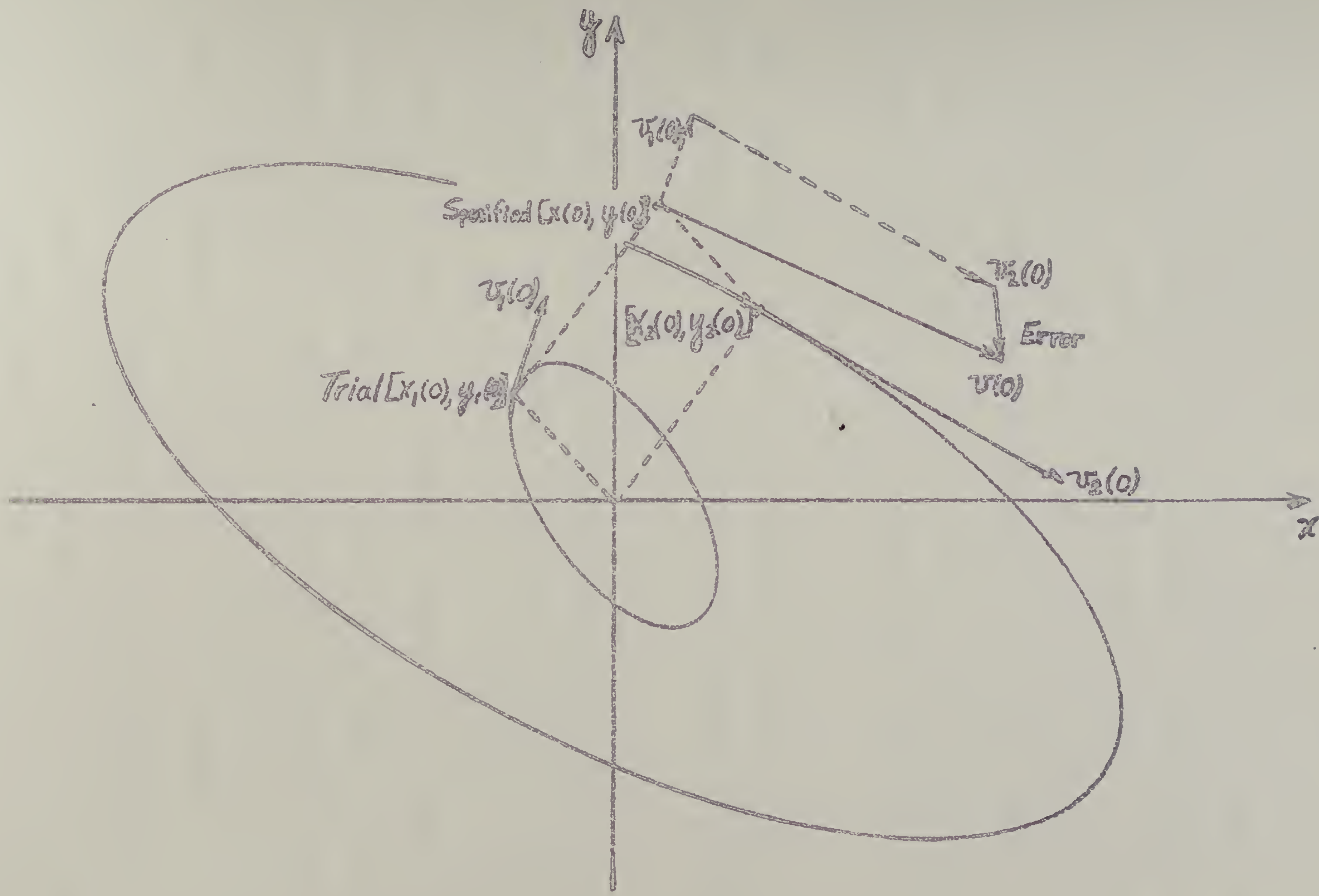


Fig. 5.1. Determination of Orbit from Specified Initial Conditions

5.0 Desirable Ranges of Initial Conditions

From the preceding analysis, it appears that, given an unavoidable displacement error, we will obtain the "smoothest" orbit if the initial velocity has a magnitude in a range such that the Jacobian constant of the vehicle is approximately the same as that of a body at rest at the Trojan point. It can be seen by substitution in Eq. 4.19 that a body vibrating in the p_1 mode only has a relative energy higher than that of a body at rest at the Trojan point, while a body vibrating in the p_2 mode only has a relative energy lower than that of a body at rest at the Trojan point.

More important, we wish the initial velocity to be in a direction compatible with a clockwise orbit about the Trojan point (using the notation of Fig. 4.1).

The method of the preceding section shows that, for initial velocities in the "correct" direction, we tend to get contributions from the two modes which are additive, and the outer bound of the resulting orbital motion is not much different from the initial displacement. On the other hand, if the initial velocity is in the opposite direction, we tend to get large contributions of the two modes which subtract at the initial point, but add at the outer bound of the orbit.

This verifies the impression we get from section 4.3. Since the gravitational and centrifugal force field is, in general, directed away from the Trojan point, the Coriolis acceleration

must provide the net acceleration toward the Trojan point for a stable orbit. For this to occur, the gross motion of the vehicle must be clockwise as can be seen in Fig. 4-2.

Fig. 5-2, which is also not drawn to scale, is a sketch of two attempted injections of a vehicle into a Trojan point orbit. In each case the approach is from the negative x direction, and the vehicle has been provided with approximately the correct relative energy. The closest point of approach to the Trojan point is considered the initial point for generating the orbit.

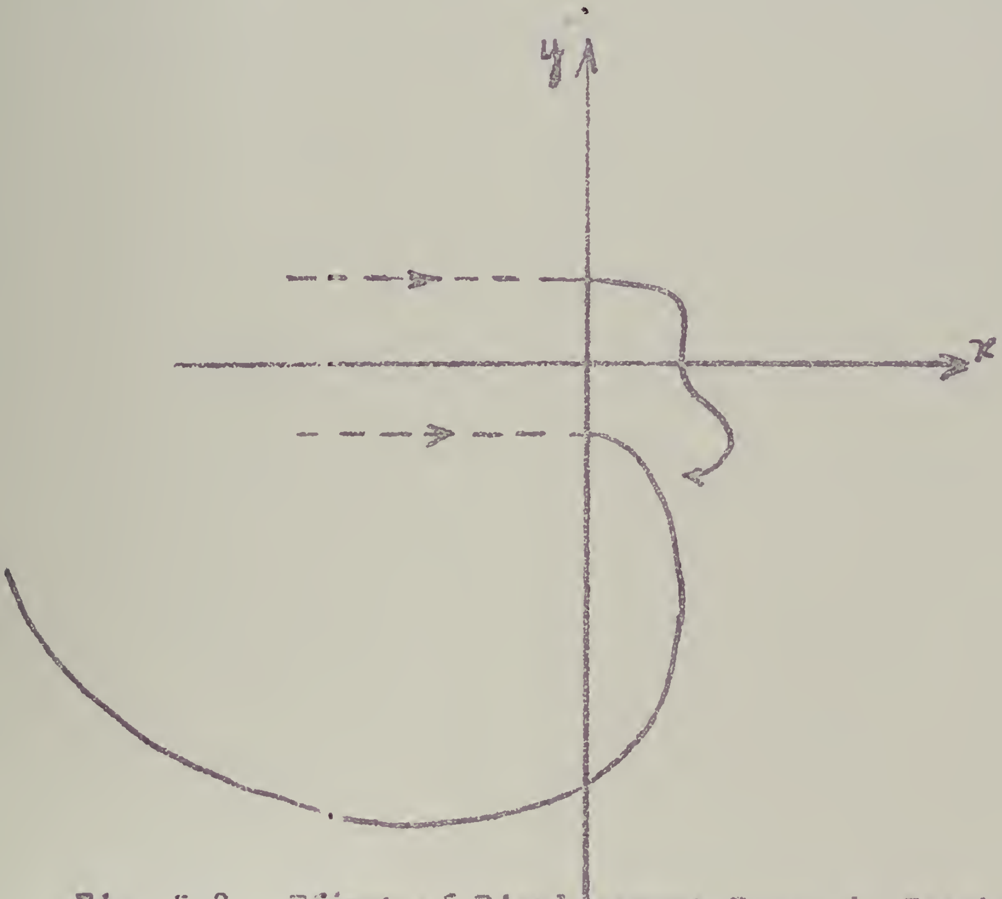


Fig. 5.2 Effect of Displacement Error in Establishing a Trojan Point Orbit

Under these conditions, a miss to the left of the Trojan point results in an orbit whose maximum displacement from the Trojan point is much smaller than that of the orbit resulting from a miss to the right of the Trojan point. In an actual attempted injection of a vehicle into this type of orbit it appears we should evaluate the expected angular error in guidance, and then aim to the side which will insure a resulting orbit of the first type.

CHAPTER 6

COMPARISON OF THE ACTUAL EQUATIONS OF MOTION WITH THE LINEARIZED EQUATIONS

We will define, $\bar{F}^*$, the residual force, to be the vector difference between the actual force exerted on a vehicle by the combined gravitational and centrifugal force, and the fictitious force which would be exerted on the vehicle by the linearized model of the system.

$$\bar{F}^* = \bar{F} - \bar{F}' = \text{grad } u - \text{grad } u' = \text{grad } u^*$$

where

$$u^* = u - u'$$

and the quantities u and u' are defined by equations 4.5 and 4.6.

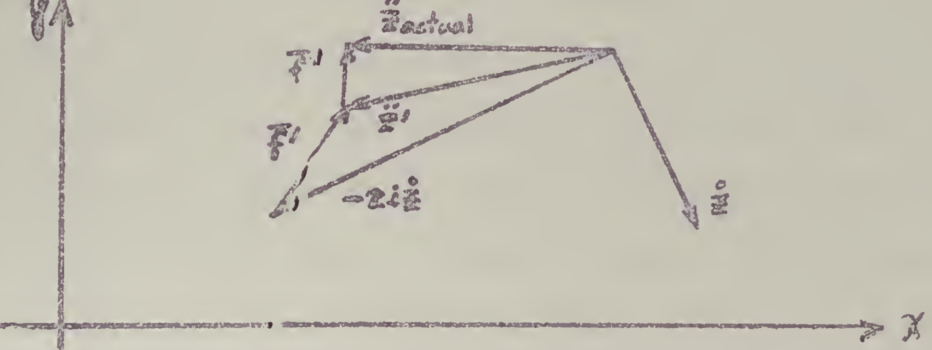


Figure 6.1. Effect of the Residual Force on the Acceleration of the Vehicle

The residual force directly represents the effect of all higher order terms of the equations of motion which were neglected in their linearization. As shown in Fig.6.1, if the vehicle is assumed to follow the solution of the linearized equations as a first approximation, then the residual force vector shows the tendency of the actual system to cause the vehicle to deviate from this path.

It would be convenient if the residual force field were found to have some significant pattern. For example, if a solution of the linearized equations was formed where the residual force was always inward toward the Trojan point, this would be an indication of the existence of a stable orbit somewhere within this region.

Several points of the residual Jacobian function, u^*, were calculated by the methods shown in Appendix B. The results of these calculations are shown in Fig. 5.1. The residual force is the gradient of the function u^*, i.e., it acts along the direction of greatest rate of increase of u^* with distance, and its magnitude is equal to that rate of increase. The plot of Fig. B.1 is neither sufficiently complete nor accurate enough for a numerical determination of the residual force. In any case, it is more easily calculated directly by components with the methods of Appendix B. Instead the plot was meant to give a rough indication of the direction and magnitude of the residual force as a function of displacement from the Trojan point.

No significant pattern of residual force distribution was found. If we trace out an orbit about the Trojan point, we find that the residual force flunctuates in magnitude and direction with no readily observable net result. We also see that the closer we are to the Trojan point, the smaller is the residual force.

So far, we have found nothing to indicate that our linearized model is not a good approximation of the actual system. The deviations show no significant patterns, and they seem to decrease regularly as we get closer to the point about which the equations were linearized. However, we are still not guaranteed that an orbit about the Trojan point is absolutely stable for "small disturbances", nor do we have any information about how small the disturbances must be to obtain "partial stability," i.e., keeping a vehicle within specified limits of displacement from the Trojan point for a specified period of time.

CHAPTER 7

THE PROBLEM OF DETERMINING STABILITY BY DIGITAL COMPUTATION OF THE ORBIT

Usually when we seek an analytical solution of a system of non-linear differential equations, we feel that we can always fall back on numerical method is all else fails. We would like to have a solution in closed form or, in this case, an analytical upper bound on the orbit, but, in general, this is a convenience which saves us the time and expense of resorting to a digital computer. Here, there is some doubt about whether we can obtain a satisfactory answer to the stability question by numerical methods.

The obvious method is merely to integrate the equations of motion numerically over as long a time interval as is required, and thus obtain a numerical representation of the position of the vehicle throughout the specified time interval. The limitations on this method fall into two classes.

The first limitation is our inaccurate knowledge of the physical constants of the system. Although it is believed that our estimates of the earth-moon mass ratio may be in error by as much as 0.3 per cent, our present linar rocket experiments should significantly advance our knowledge in this area. In

any case, this is not a basic limitation of the investigation of stability. If we can obtain the answers to the stability questions based on a model of the earth-moon system where $\frac{m_2}{m_1}$ is exactly 0.01213, we will have taken a long step toward obtaining answers for the actual system. If later, with more precise knowledge of the mass ratio, we feel that the results do not apply to the actual system, the procedures for computation will still apply.

The major limitation on determining a precision orbit is the errors inherent in the method of computation. The effect of the choice of a particular method of computation is discussed by Baker and Others (6). Basically, these errors may be classified as truncation error and rounding error. Truncation error results from the use of a finite interval over which the value of the integrand is approximated. It can be reduced by decreasing the size of the interval, and thus more closely approximating the value of the integrand over the range of integration. Rounding error is due to the limitation of computation to a finite number of digits, and the error in each step of the computation is propagated throughout the entire problem. The resulting rounding error then increases with the number of steps.

These errors can be reduced by intelligent selection of methods of computation. For example, for an orbit of this type, it is felt that computation of perturbations from a reference orbit would be more efficient than direct integration of the

equations. However, the combination of truncation error and rounding error still can not be eliminated. For any precision orbit computation, there is a practical limit to the duration of the orbit which we can compute without a loss of validity from the cumulative error. Thus, for a general lunar probe, regardless of refinements in the method of computation, we can specify where it will be on its first pass near the moon, we can estimate where it will be a few months later, but we have no idea where it will be several years later.

Intuitively, it is felt that we can do better than this with a periodic orbit such as ours. It seems that, if the orbit is periodic, we can compute a precision orbit to complete one revolution about the Trojan point, and the orbit is stable if it passes closer to the Trojan point the second time around. But we have neglected the motion of the vehicle in two modes with incommensurable periods. A hypothetical vehicle moving in the linearized system will or will not meet the above criterion according to its initial conditions, but we know that the orbit of this vehicle is stable. It remains within upper and lower bounds determined by the initial conditions, and will pass arbitrarily close to any point in this region if we wait long enough.

The next thought is that we should then compare the computed precision orbit with the solution of the linearized equations. But again this comparison gives no conclusive evidence of stability. At any comparison point, the motion of the vehicle in

the actual system will have altered the proportions of relative energy between the two modes. We may find that the actual orbit is closer to the Trojan point, but if we compute a new linearized orbit using an initial conditions the actual displacement and velocity at the comparison point, we may find that the predicted upper bound of subsequent displacement from the Trojan point is greater than that predicted by the first linearized orbit.

We can, of course calculate the predicted upper bound and continue, watching this upper bound for an increase or decrease, But now we are back to the problem of not being able to watch it long enough, because of the cumulative error of the numerical calculations. It is this cumulative error that may mask the stability effect that we wish to determine, especially if the instability of the orbit is small. We are working in a region of cislunar space roughly bounded by two ellipses defined by the solution of the linearized equations. Since we see no significant pattern in the distribution of the residual forces (or non-linear effects) in this region, we should cover the entire region rather uniformly in our calculations. Otherwise, we cannot be sure that the next few revolutions of the vehicle about the Trojan point will not pass through points where the non-linear effects are entirely different.

The length of time necessary for an orbit to pass within a specified distance, δ, from every point in its permitted region varies with the distribution of the motion between the two modes

of periodic motion. If the motion of the vehicle is primarily confined to one mode with the other of amplitude no greater than δ, then the necessary time is no longer than the period, of the primary mode. If the amplitude of the secondary mode is large, the time may be prohibitive due to cumulative error.

This leads us to consider investigation of the special cases where, based on the linearized model of the system, the initial conditions are such as to excite only one mode of vibration. The trouble with this is that we should expect the non-linearities of the actual system to cause transfer of energy to the other mode, with a cumulative effect until we are back with the general case again.

It seems possible that we may obtain useful information by constraining the orbit to the simple ellipse, and integrating the effect of the residual force over one revolution about the Trojan point. Intuitively, it is felt that a net inward force would be an indication of stability while a net outward force would indicate instability. If this were done for orbits of various sizes, we would then have an indication of stability versus average distance from the Trojan point. This investigation has disclosed no mathematical justification for asserting the meaningfulness of this "stability indication". However, it is felt that the discovery of a valid stability criterion will require calculations of this type.

CHAPTER 8

LINEARIZATION OF THE EQUATIONS OF MOTION AT POINTS DISPLACED FROM THE TROJAN POINT

8.1 Linearization of the Equations of Motion at the Position of the Vehicle

In a digital computation of the orbit of a vehicle about a Trojan point, it seems most efficient to use a reference orbit and compute deviations from this orbit. For a reference orbit, the first choice that comes to mind is the solution of the linearized equations as developed in Chapter 5. However, for a small segment of the orbit, we should get a better fitting reference orbit by linearizing the equations of motion at a point on the orbit, rather than at the Trojan point.

Linearizing equations 4.3 and 4.4 at the point (a,b) rather than at the point (0,0), the Taylor expansions are

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}(a,b) + \frac{\partial^2 u}{\partial x^2}(a,b)(x-a) + \frac{\partial^2 u}{\partial x \partial y}(a,b)(y-b) \tag{8.1}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}(a,b) + \frac{\partial^2 u}{\partial x \partial y}(a,b)(x-a) + \frac{\partial^2 u}{\partial y^2}(a,b)(y-b) \tag{8.2}$$

For convenience in writing the equations, let R_1 and R_2 be the distances from the point (a,b) to the earth and moon respectively. Then from Figure 4.1

$$R_1^2 = \left(b + \frac{\sqrt{3}}{2}\right)^2 + \left(a + \tfrac{1}{2}\right)^2 = b^2 + \sqrt{3}b + a^2 + a + 1 \qquad (8.3)$$

$$R_2^2 = \left(b + \frac{\sqrt{3}}{2}\right)^2 + \left(a - \tfrac{1}{2}\right)^2 = b^2 + \sqrt{3}b + a^2 - a + 1 \qquad (8.4)$$

Equations 8.1 and 8.2 can be written as

$$\frac{\partial u}{\partial x} = k_1 + k_2\,(x-a) + k_3\,(y-b) \qquad (8.5)$$

$$\frac{\partial u}{\partial x} = k_4 + k_3\,(x-a) + k_5\,(y-b) \qquad (8.6)$$

where, from equations 4.17 through 4.19, with substitution of equations 8.3 and 8.4

$$k_1 = \frac{\partial u}{\partial x}(a,b) = -\epsilon + a + \tfrac{1}{2} - \lambda\left(a+\tfrac{1}{2}\right) R_1^{-3} - \epsilon\left(a-\tfrac{1}{2}\right) R_2^{-3} \qquad (8.7)$$

$$k_2 = \frac{\partial^2 u}{\partial x^2}(a,b) = 1 - \lambda R_1^{-3} - \epsilon R_2^{-3} + 3\lambda\left(a+\tfrac{1}{2}\right)^2 R_1^{-5} + 3\epsilon\left(a-\tfrac{1}{2}\right)^2 R_2^{-5} \qquad (8.8)$$

$$k_3 = \frac{\partial^2 u}{\partial x \partial y}(a,b) = 3\lambda\left(a+\tfrac{1}{2}\right)\left(b+\frac{\sqrt{3}}{2}\right) R_1^{-5} + 3\epsilon\left(a-\tfrac{1}{2}\right)\left(b+\frac{\sqrt{3}}{2}\right) R_2^{-5} \qquad (8.9)$$

$$k_4 = \frac{\partial u}{\partial y}(a,b) = b + \frac{\sqrt{3}}{2} - \lambda\left(b + \frac{\sqrt{3}}{2}\right) R_1^{-3} - \epsilon\left(b + \frac{\sqrt{3}}{2}\right) R_2^{-3} \quad (8.10)$$

$$k_5 = \frac{\partial^2 u}{\partial y^2}(a,b) = 1 - \lambda R_1^{-3} - \epsilon R_2^{-3} + 3\lambda\left(b + \frac{\sqrt{3}}{2}\right)^2 R_1^{-5} + 3\epsilon\left(b + \frac{\sqrt{3}}{2}\right) R_2^{-5} \quad (8.11)$$

In linearized form, equations 4.3 and 4.4 can be written as

$$\ddot{x} - 2\dot{y} = (k_1 - ak_2 - bk_3) + k_2 x + k_3 y \quad (8.12)$$

$$\ddot{y} + 2\dot{x} = (k_4 - ak_3 - bk_5) + k_3 x + k_5 y \quad (8.13)$$

These equations can be written in differential operator form as

$$(p^2 - k_2)\,x - (2p + k_3)\,y = k_1 - ak_2 - bk_3$$

$$(2p - k_3)\,x + (p^2 - k_5)\,y = k_4 - ak_3 - bk_5$$

The characteristic equation of this system is

$$(p^2 - k_2)(p^2 - k_5) + (2p + k_3)(2p - K_3) = 0$$

or

$$p^4 + (4 - k_2 - k_5)\,p^2 + (k_2 k_5 - k_3^2) = 0$$

and the roots of the characteristic equation are given by

$$p^2 = \frac{-4-k_2-k_5}{2} \pm \sqrt{\frac{(4-k_2-k_5)^2}{4} - (k_2 k_5 - k_3^2)}$$

A "stable" solution is obtained if

$$\frac{(4-k_2-k_5)^2}{4} - k_2k_5 + k_3^2 > 0$$

Substituting for the k's from equations 8.7 through 8.11, the stability criterion is

$$\frac{9}{4}\left(\frac{\lambda}{R_1^3} + \frac{\epsilon}{R_2^3}\right)^2 - 2\left(\frac{\lambda}{R_1^3} + \frac{\epsilon}{R_2^3}\right) - \frac{9\epsilon\lambda\left(b + \frac{\sqrt{3}}{2}\right)^2}{R_1^5\, R_2^5} > 0 \tag{8.14}$$

Equations 8.12 and 8.13 can be solved to obtain a solution in the form of constants plus pure sinusoids if the criterion of Eq. 8.14 is met. If the criterion is not met, the solution contains exponentially increasing sinusoids.

Whether or not any significance can be attached to this criterion is not immediately apparent. If the solution resulting from a linearization of the equations about a point displaced from the Trojan point is to be considered better than the solution of Chapter 5, it can be used only over a small segment of the orbit, and then a linearization in the vicinity of a new point must be accomplished. We cannot then categorically state that the stability of a specific periodic solution is significant if we consider only a small fraction of one period of the solution.

On the other hand, it is felt that, if the vehicle passes through a region where the solution of its linearized equations

is unstable, there should be a tendency toward stability of the complete orbit. It seems that each time the vehicle passes through this region, there would be a cumulative tendency for it to be pushed further away from the Trojan point.

Time was not available for a survey of the region in the vicinity of the Trojan point to determine where the criterion of Eq. 8.14 is met. Roughly it seems that it will not be met for either x or y greater than about 0.1. In any case, we should first have a better justification of the significance of such a survey. Although a time limit on the present study precluded further investigation along these lines, it is felt that there may be something useful in this area.

CHAPTER 9

OTHER METHODS OF ORBITAL DETERMINATION

At this point, the ignorance of the author of the intricacies of classical celestial mechanics is probably already apparent. The literature has been searched, but not thoroughly digested. If the method by which our problem will be solved remains hidden in this field, it can most probably be found in the work of Aurel Wintner (7).

The recent Russian work on the subject has not been thoroughly examined by the author, partly because of the non-availability of translations. The work of Merman (8) is briefly discussed by Leimanis (9). This method seems to be also described by an abstract of a paper by Schigolev (10):

"The motion of an asteroid is calculated according to approximation formulas..... The approximation values desired are obtained by averaging. The average method of the author, however, differs from the well-known method of celestial mechanics inasmuch as the author does not average the power function, but certain skillfully chosen parts of the differential equations and thus obtains solvable differential equations".

This method was investigated far enough to ascertain that periodic solutions were indeed obtained, but not far enough to judge the validity of these solutions.

CHAPTER 10

CONCLUSIONS

As stated in Chapter 1, we do not have specific answers to the problem under consideration. If we are asked "How can the orbital stability of a proposed Trojan point satellite be predicted?", we can suggest the following approach:

1. Evaluate the expected guidance inaccuracy in terms of velocity and displacement error at the Trojan point. Select the aiming point as suggested in Section 5.6 to minimize the size of the resulting orbit about the Trojan point, based on the solution of the linearized equations.

2. Compute the orbit around the Trojan point, using as incremental reference orbits, solutions of the equations of Chapter 8. The length of the increments used in the computation will be picked to minimize the combination of truncation error and rounding error. Also an attempt should be made to predict how the size of this error will increase with time. If the computed orbit diverges from the Trojan point before this computation error becomes significant, the question is answered.

But a correct negative answer is much easier to obtain than a correct positive answer. If the computed orbit diverges from the Trojan point only slightly, it will not be clear whether

this is due to an actual instability of the orbit, or a result of the accumulated computation error.

3. Further work on the orbital stability of Trojan satellites should probably be directed toward the problem of identifying and reducing the effect of computational errors. Also it is felt that further investigation should be made of the significance of the region in which stable solutions of the linearized equations are obtained, as discussed in Chapter 8.

APPENDIX A

NUMERICAL SOLUTION OF THE LINEARIZED EQUATIONS OF MOTION

The following calculations were made to permit calculation of a solution of the linearized equations directly from specified initial conditions. For numerical work, a value of ϵ of exactly 0.01213 was used. Although, the physical quantity in the earth-moon system is only known to within about 0.3 per cent, the calculations were carried to ten significant figures to permit use of the results in subsequent numerical integration computations, if desired. The results are then consistent with a linearized model of the system with the mass ratio specified by $\epsilon = 0.01213$ exactly.

$$\lambda = 1 - \epsilon = 0.98787$$

$$\sqrt{1-27\epsilon(1-\epsilon)} = 0.8224735232$$

$$p_1^2 = \tfrac{1}{2}(1 + 0.8224735232) = 0.9112367616$$

$$p_2^2 = \tfrac{1}{2}\,(1-0.8224735232) = 0.0887632384$$

$$p_1 = 0.9545872205$$

$$p_2 = 0.2979316002$$

$$\frac{3\sqrt{3}}{4}\,(1-2\epsilon) = 1.267523442$$

Substituting initial conditions $x\,(0)$, $y\,(0)$, $\dot{x}\,(0)$ and $\dot{y}\,(0)$ into equations 5.1 and 5.2

$$x\,(0) = B_1 + D_1 \qquad \text{(A.1)}$$

$$y\,(0) = B_2 + D_2 \qquad \text{(A.2)}$$

$$\dot{x}\,(0) = p_1 A_1 + p_2 C_1 \qquad \text{(A.3)}$$

$$\dot{y}\,(0) = p_1 A_2 + p_2 C_2 \qquad \text{(A.4)}$$

Combining equations 5.3 and 5.4 we can get A_1 in terms of B_1 and B_2

$$2p_1A_1 = \frac{3\sqrt{3}}{4}(1-2\epsilon)\,B_1 + \left(p_1^2 + \frac{9}{4}\right)B_2 \qquad \text{(A.5)}$$

Similarly from equations 5.5 and 5.6

$$2p_2C_1 = \frac{3\sqrt{3}}{4}(1-2\epsilon)\,D_1 + \left(p_2^2 + \frac{9}{4}\right)D_2 \qquad \text{(A.6)}$$

From equations A.1 and A.2

$$D_1 = x(0) - B_1 \qquad \text{(A.7)}$$

$$D_2 = y(0) - B_2 \qquad \text{(A.8)}$$

Substituting equations A.7 and A.8 into Eq. A.6

$$2p_2C_1 = \frac{3\sqrt{3}}{4}(1-2\epsilon)\left[x(0) - B_1\right] + \left(p_2^2 + \frac{9}{4}\right)\left[y(0) - B_2\right] \qquad \text{(A.9)}$$

Multiplying Eq. A.3 by 2 and substituting from equations A.5 and A.9

$$2\dot{x}(0) = \frac{3\sqrt{3}}{4}(1-2\epsilon)\left[B_1 + x(0) - B_1\right] + \left(p_1^2 + \frac{9}{4}\right)B_2 + \left(p_2^2 + \frac{9}{4}\right)\left[y(0) - B_2\right] \qquad \text{(A.10)}$$

$$2\dot{x}(0) = \frac{3\sqrt{3}}{4}(1-2\epsilon)\, x(0) + (p_1^2 - p_2^2)\, B_2 + (p_2^2 + \frac{9}{4})\, y(0)$$

(A.11)

$$B_2 = \frac{\frac{3\sqrt{3}}{4}(1-2\epsilon)}{p_1^2\ p_2^2}\, x(0) - \frac{p_2^2 + \frac{9}{4}}{p_1^2 - p_2^2}\, y(0) + \frac{2}{p_1^2 - p_2^2}\, \dot{x}(0)$$

(A.12)

Substituting numerical values

$$B_2 = -1.541111544\, x(0) - 2.843572677\, y(0) + 2.431689219\, \dot{x}(0)$$

(A.13)

Combining equations 5.3 and 5.4 to eliminate A_1 instead of A_2, we can get A_2 in terms of B_1 and B_2

$$2p_1A_2 = -(p_1^2 + \frac{3}{4})\, B_1 - \frac{3\sqrt{3}}{4}(1-2\epsilon)\, B_2 \quad (A.14)$$

Similarly from equations 5.5 and 5.6

$$2p_2C_2 = -(p_2^2 + \frac{3}{4})\, D_1 - \frac{3\sqrt{3}}{4}(1-2\epsilon)\, D_2 \quad (A.15)$$

Substituting equations A.7 and A.8 into Eq. A.15

$$2p_2C_2 = -\left(p_2^2 + \frac{3}{4}\right)\left[x(0) - B_1\right] - \frac{3\sqrt{3}}{4}(1-2\epsilon)\left[y(0) - B_2\right] \tag{A.16}$$

Multiplying Eq. A.4 by 2 and substituting from equations A.14 and A.16

$$2\dot{y}(0) = -\left(p_1^2 + \frac{3}{4}\right)B_1 - \frac{3\sqrt{3}}{4}(1-2\epsilon)\left[B_2 + y(0) - B_2\right] - \left(p_2^2 + \frac{3}{4}\right)\left[x(0) - B_1\right] \tag{A.17}$$

$$2\dot{y}(0) = -\frac{3\sqrt{3}}{4}(1-2\epsilon)\,y(0) - (p_1^2 - p_2^2)B_1 - \left(p_2^2 + \frac{3}{4}\right)x(0) \tag{A.18}$$

$$B_1 = -\frac{p_2^2 + \frac{3}{4}}{p_1^2 - p_2^2}\,x(0) - \frac{\frac{3\sqrt{3}}{4}(1-2\epsilon)}{p_1^2 - p_2^2}\,y(0) - \frac{2}{p_1^2 - p_2^2}\,\dot{y}(0) \tag{A.19}$$

Substituting numerical values

$$B_1 = -1.019805762\,x(0) - 1.541111544\,y(0) - 2.431689219\,\dot{y}(0) \tag{A.20}$$

A_1 can now be found from Eq. A.5

$$A_1 = \frac{\frac{3\sqrt{3}}{4}(1-2\epsilon)}{2p_1} B_1 + \frac{p_1^2 + \frac{9}{4}}{2p_1} B_2 \qquad (A.21)$$

$$A_1 = 0.6639118009\ B_1 + 1.655813473\ B_2 \qquad (A.22)$$

Substituting from equations A.20 and A.13

$$A_1 = 3.228854338\ x(0) - 5.731588091\ y(0)$$

$$+4.026423771\ \dot{x}(0) - 1.614427169\ \dot{y}(0)$$

$$(A.23)$$

Similarly from Eq. A.14

$$A_2 = -\ \frac{p_1^2 + \frac{3}{4}}{2p_1} B_1 - \frac{\frac{3\sqrt{3}}{4}(1-2\epsilon)}{2p_1} B_2$$

$$(A.24)$$

$$A_2 = -0.8701335645\ B_1 - 0.6639118009\ B_2$$

$$(A.25)$$

Substituting from equations A.20 and A.13

$$A_2 = 1.910529363\, x(0) + 3.228854338\, y(0)$$

$$-1.614427169\, \dot{x}(0) + 2.115894408\, \dot{y}(0) \qquad (A.26)$$

Substituting Eq. A.20 into Eq. 4.7

$$D_1 = 2.019805762\, x(0) + 1.541111544\, y(0) + 2.431689219\, \dot{y}(0)$$

(A.27)

Substituting Eq. A.13 into Eq. A.8

$$D_2 = 1.541111544\, x(0) + 3.843572677\, y(0) - 2.431689219\, \dot{x}(0)$$

(A.28)

C_1 can be found from Eq. A.6

$$C_1 = \frac{\frac{3\sqrt{3}}{4}(1-2e)}{2p_2} D_1 + \frac{p_2^2 + \frac{9}{4}}{2p_2} D_2$$

(A.29)

$$C_1 = 2.127205440\, D_1 + 3.925000296\, D_2 \qquad (A.30)$$

Substituting from equations A.27 and A.28

$$C_1 = 10.34540507\ x(0) + 18.36428476\ y(0)$$

$$-9.544380904\ \dot{x}(0) + 5.172702535\ \dot{y}(0) \tag{A.31}$$

Similarly from Eq. A.15

$$C_2 = -\frac{p_2^2 + \frac{3}{4}}{2p_2}\ D_1 - \frac{\frac{3\ 3}{4}\ (1-2\epsilon)}{2p_2}\ D_2 \tag{A.32}$$

$$C_2 = -\ 1.407643965\ D_1 - 2.127205440\ D_2 \tag{A.33}$$

Substituting from equations A.27 and A.28

$$C_2 = -\ 6.121428251\ x(0) - 10.34540507\ y(0)$$

$$+ 5.172702535\ \dot{x}\ (0) - 3.422952654\ \dot{y}(0) \tag{A.34}$$

Given specified initial conditions, the constants of integration can now be calculated from equations A.13, A.20, A.23, A.25, A.27, A.28, A.31 and A.34. These equations were cross-checked numerically by other formulas and found to be consistent.

The substitution of numbers into this equation tells us little about what sort of an orbit we would get with slightly different initial conditions, as does the method of Section 5.5. However, for calculation of a specific orbit, the numerical work is much simpler. If this solution were required as a reference orbit for a digital computation of the orbit, the above equations would be used.

APPENDIX B

NUMERICAL COMPUTATION OF THE RESIDUAL JACOBIAN FUNCTION

To determine some numerical values of u and u',
equations 4.5 and 4.20 were transformed to polar coordinates by the equations

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta$$

The equations then become

$$u = - \epsilon \rho \cos\theta + \tfrac{1}{2} (\rho^2 \sin^2\theta + \sqrt{3}\rho \sin\theta + \rho^2 \cos^2\theta + \rho \cos\theta + 1)$$

$$+ \lambda (\rho^2 \sin^2\theta + \sqrt{3}\rho \sin\theta + \rho^2 \cos\theta + \rho\cos\theta + 1)^{-\frac{1}{2}}$$

$$+ \epsilon (\rho^2 \sin^2\theta + \sqrt{3}\rho \sin\theta + \rho^2 \cos^2\theta - \rho\cos\theta + 1)^{-\frac{1}{2}} \tag{B.1}$$

$$u' = \frac{3}{2} + \frac{3}{8} \rho^2 \cos^2\theta + \frac{3\sqrt{3}}{4} (1-2\epsilon) \rho^2 \sin\theta \cos\theta + \frac{9}{8} \rho^2 \sin^2\theta \tag{B.2}$$

Using the identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\frac{\sqrt{3}}{2}\sin\theta + \tfrac{1}{2}\cos\theta = \sin 60^\circ \sin\theta + \cos 60^\circ \cos\theta =$$

$$\cos\ (\theta - 60^\circ)$$

$$\frac{\sqrt{3}}{2}\ \sin\theta - \tfrac{1}{2}\cos\theta = -\cos\ (\theta + 60^\circ)$$

The equations for u and u' become

$$u = -\epsilon\rho\cos\theta + \tfrac{1}{2}\left[\rho^2 + 2\rho\cos\ (\theta - 60^\circ) + 1\right]$$

$$+\lambda\left[\rho^2 + 2\rho\cos\ (\theta - 60^\circ) + 1\right]^{-\frac{1}{2}} + \epsilon\left[\rho^2 - 2\rho\cos(\theta + 60^\circ) + 1\right]^{-\frac{1}{2}}$$

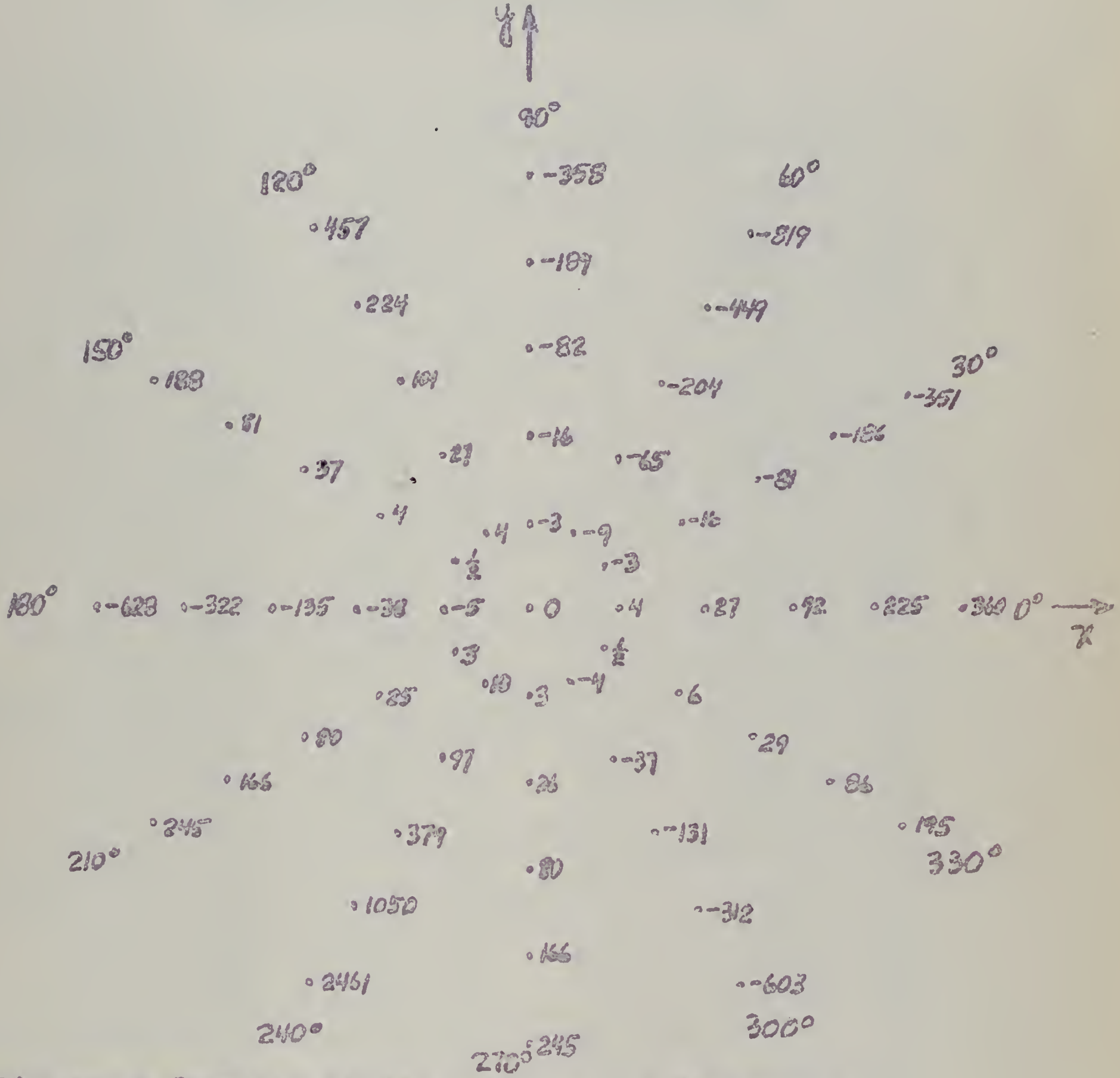
(B.3)

$$u' = \frac{3}{2} + \frac{3}{4}\rho^2 + \frac{3\sqrt{3}}{8}(1-2\epsilon)\rho^2\sin 2\theta - \frac{3}{8}\rho^2\cos 2\theta$$

(B.4)

Using a value for ϵ of 0.01213, the quantities u and u' were calculated for 0.1 increments of   from 0 to 0.5 and for

30° increments of θ. The results are listed in Table B.1. The residual function $u^* = u - u'$ at these points is shown in Fig. B.1.



Distances from the Trojan point in 0.1 increments

Indicated values are $u^* \times 10^4$

Fig. B.1. Approximate Values of the Residual Jacobian Function

TABLE B.1

TABULATION OF JACOBIAN FUNCTIONS

θ	ρ	u	u'	θ	ρ	u	u'
0°	.1	1.5041	1.5038	180°	11	1.5033	1.5038
	.2	1.5177	1.5150		.2	1.5112	1.5150
	.3	1.5429	1.5338		.3	1.5203	1.5338
	.4	1.5825	1.5600		.4	1.5278	1.5600
	.5	1.6297	1.5938		.5	1.5310	1.5938
30°	.1	1.5108	1.5111	210°	.1	1.5114	1.5111
	.2	1.5429	1.5445		.2	1.5470	1.5445
	.3	1.5919	1.6000		.3	1.6080	1.6000
	.4	1.6592	1.6778		.4	1.6944	1.6778
	.5	1.7427	1.7778		.5	1.8023	1.7778
60°	.1	1.5140	1.5149	240°	.1	1.5159	1.5149
	.2	1.5529	1.5595		.2	1.5692	1.5595
	.3	1.6134	1.6338		.3	1.6717	1.6338
	.4	1.6930	1.7338		.4	1.8428	1.7378
	.5	1.7927	1.8716		.5	2.1177	1.8716
90°	.1	1.5109	1.5113	270°	.1	1.6116	1.5113
	.2	1.5434	1.5450		.2	1.5476	1.5450
	.3	1.5930	1.6013		.3	1.6093	1.6013
	.4	1.6611	1.6800		.4	1.6966	1.6800
	.5	1.7455	1.7813		.5	1.8058	1.7813
120°	.1	1.5043	1.5039	300°	.1	1.5035	1.5039
	.2	1.5183	1.5156		.2	1.5119	1.5156
	.3	1.5441	1.5350		.3	1.5219	1.5350
	.4	1.5846	1.5622		.4	1.5310	1.5622
	.5	1.6329	1.5972		.5	1.5369	1.5972
150°	.1	1.5002	1.5001	330°	.1	1.5002	1.5001
	.2	1.5010	1.5006		.2	1.5012	1.5006
	.3	1.5049	1.5012		.3	1.5041	1.5012
	.4	1.5103	1.5022		.4	1.5108	1.5022
	.5	1.5222	1.5034		.5	1.5229	1.5034

[illegible]

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[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
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REFERENCES

1. Bucheim, R.W., "Motion of a Small Body in Earth-Moon Space," The Rand Corporation, Research Memorandum RM-1726.

2. Klemperer, W.B., and Benedikt, E.T., "Synodic Satellites," Astronautics, Vol. 3, No. 10, p. 28, October, 1958.

3. Moulton, F.R., An Introduction to Celestial Mechanics, 2nd rev. ed., Macmillan, N.Y., 1958.

4. Whittaker, E.T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, With an Introduction to the Problem of Three Bodies, 4th ed., Dover, N.Y., 1944.

5. Bucheim, R.W., "Lunar Flight Trajectories," Space Technology, ed. H.S. Seifert, John Wiley & Sons, N.Y., 1959, Ch. 7.

6. Baker, R.M.L, and Others, "Efficient Precision Orbit Computational Techniques," AFOSR TN-59-766, 1959.

7. Wintner, A., The Analytical Foundations of Celestial Mechanics, Princeton University Press, Princeton, N.J., 1947.

8. Merman, G.A., "A New Class of Periodic Solutions in the Restricted Three-Body Problem and Hill's Problem," Trudy Inst. Teor. Astr., 1, 5-86 (1952), trans. C.T. Benster, UCLA.

9. Leimanis, E. and Minorsky, N., *Dynamics and Nonlinear Mechanics*, John Wiley & Sons, N.Y., 1958, 55-57.

10. Schigolev, B.M., "On the Approximate Calculation of the Ephemerides inthe Restricted Three-Body Problem," Vestnik Moskovskogo universiteta, Seriya matematiki, mekhaniki, astonomii, fiziki, khimii, 1958, Nr.5,pp. 37-48 (USSR).

11. Lagerstrom, P.A., and Graham, M.E., "Some General Concepts of Dynamics and Their Application to the Restricted Three-Body Problem," Douglas Report No. SM-23401, 1958.

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